Primes matrix: Approximation

https://blog.carolin-zoebelein.de/2018/03/primes-matrix-approximation.html
Sat 31 Mar 2018 in Math, Carolin Zöbelein

Last week, I wrote in Primes matrix about a possibility to represent primes. The negative part of this was, that there isn't a nice way to generate $X_{(i)}^{n\times n}$. Today, I want to add a few lines, how you can approximate it.

Let's look again at

$$X_{(1)}^{n\times n} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(x_{(1),kj}\right)_{k=1,\dots,n\ ,j=1,\dots,n}\ \delta_{kj}$$

$$X_{(2)}^{n\times n} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \left(x_{(2),kj}\right)_{k=1,\dots,n\ ,j=1,\dots,n}\ \delta_{kj}$$

Instead of the definition from my last post, now, we will define $x_{(i),kj}$ in the following way

$$x_{(i),kj} := \lim_{m\to\infty} \cos^{2m}\left(2\pi \frac{k-x_i}{2x_i+1}\right)\delta_{kj}$$

and

$$\delta_{kj} := \begin{cases} 1 & \text{if } k=j \\ 0 & \text{else} \end{cases}$$

Explanation: Like in our old definition, we always get an '1' at all places which presents a divisible number for a given x_i. Now, at the other positions, we get an value v between $0 < |v| < 1$, instead of '0' in our old definition. To get our '0's, like before, we use $\lim_{m\to\infty}$. The factor two ensures a positive approxiation value.

With this, we also get

$$\overline{x_{(i),kj}} := \left(1 - x_{(i),kj}\right)\delta_{kj} = \lim_{m\to\infty}\left(1 - \cos^{2m}\left(2\pi\frac{k - x_i}{2x_i + 1}\right)\right)\delta_{kj}$$

An alternative way also to write this, is given by

$$\overline{x_{(i),kj}} := \lim_{m\to\infty}\sin^{\frac{2}{m}}\left(2\pi\frac{k - x_i}{2x_i + 1}\right)\delta_{kj}$$

Explanation: We have a similiar situation like for our $x_{(i),kj}$, but in this case now, we have to use $2/m$ to get an approximation for '1'.

We see that we can use cos and also sin depending on which values we are interested in. Of course, don't forget the different cases for lim with $2m$ respectively $2/m$.

So, we can put this together to

$$\eta_{(i),kj} := \lim_{m\to\infty}\exp\left(I2\pi\frac{k - x_i}{2x_i + 1}\epsilon\left(m\right)\right)\delta_{kj}$$

I is the Imaginary unit, with $I^2 = -1$ and

$$\epsilon\left(m\right) := \begin{cases} 2m & \text{for real part } \mathrm{Re}\left(\eta\right) \\ \frac{2}{m} & \text{for imaginary part } \mathrm{Im}\left(\eta\right) \end{cases}$$

Ok. What do we need, next? We need

$$\exp\left(Iz_1\right)\cdot\exp\left(Iz_2\right) = \exp\left(I\left(z_1 + z_2\right)\right) = \cos\left(z_1 + z_2\right) + I\sin\left(z_1 + z_2\right)$$

so let's look at

$$z_1 + z_2 = 2\pi\epsilon\left(m\right)\left(\frac{k - x_1}{2x_1 + 1} + \frac{k - x_2}{2x_2 + 1}\right) = 2\pi\epsilon\left(m\right)\frac{\left(k - x_1\right)\left(2x_2 + 1\right) + \left(k - x_2\right)\left(2x_1 + 1\right)}{\left(2x_1 + 1\right)\left(2x_2 + 1\right)}$$

In https://github.com/Samdney/primescalc we already discussed a similiar situation, which showed us that we don't have any problems if $2x_1 + 1$ and $2x_2 + 1$ are primes. For more information about this, please read the mentioned paper.

So we can do our final step

$$x_{(a,\ldots,b),kj} = \lim_{m\to\infty}\left(\prod_{i=a}^{b}\exp\left(I2\pi\frac{k-x_i}{2x_i+1}\epsilon\left(m\right)\right)\right)\delta_{kj} = \lim_{m\to\infty}\exp\left(\sum_{i=a}^{b}I2\pi\frac{k-x_i}{2x_i+1}\epsilon\left(m\right)\right)\delta_{kj}$$

Now we look at the special case of $x_i = a, a+1, a+2, \ldots, b-2, b-1, b$ for the given sum

$$\sum_{i=a}^{b}I2\pi\frac{k-x_i}{2x_i+1}\epsilon\left(m\right) = \ = I2\pi\frac{1}{4}\left(2k\psi^{(0)}\left(b+\frac{3}{2}\right)+\psi^{(0)}\left(b+\frac{3}{2}\right)-2k\psi^{(0)}\left((a-1)+\frac{3}{2}\right)-\psi^{(0)}\left((a-1)+\frac{3}{2}\right.\right.$$

since

$$\sum_{x=1}^{n}\frac{k-x}{2x+1} = \frac{1}{4}\left(2k\psi^{(0)}\left(n+\frac{3}{2}\right)-2k\psi^{(0)}\left(\frac{3}{2}\right)-2k+\psi^{(0)}\left(n+\frac{3}{2}\right)-\psi^{(0)}\left(\frac{3}{2}\right)\right)$$

$\psi^{(n)}\left(x\right)$ is the n-the derivate of the digamma function.

Finally, a small comment about permitted ranges.
Since our functions work with fix period $(k-x_i)/(2x_i+1)$, we have to ignore the range $[1, x_i]$ for each x_i-equation, else we will also receive invalid results.