Cyclotomic polynomials for primes

https://blog.carolin-zoebelein.de/2018/07/cyclotomic-polynomials-for-primes.html
Wed 18 Jul 2018 in Math, Carolin Zöbelein

Currently, I'm spending my time with cyclotomic polynomials. So, I don't want to miss the possibility also to mention a small note about the connection of this polynomials to the set of prime numbers.

At first, we have a cyclotomic polynomial in the following way

$$\phi_n(x) = \prod_{\substack{1 \leq g \leq n \\ ggT(g,n)=1}} \left(x - e^{\frac{2\pi i g}{n}}\right) \tag{1}$$

with $n, g \in \mathbb{N}$, respectively

$$\begin{aligned}
x^n - 1 &= \prod_{1 \leq g \leq n} \left(x - e^{\frac{2\pi i g}{n}}\right) \\
&= \prod_{d|n} \prod_{\substack{1 \leq g \leq n \\ ggT(g,n)=d}} \left(x - e^{\frac{2\pi i g}{n}}\right) \\
&= \prod_{d|n} \phi_{n/d}(x) \\
&= \prod_{d|n} \phi_d(x)
\end{aligned} \tag{2}$$

. In the case of pime numbers $p \in \mathbb{P}$, we also have

$$\phi_p(x) = 1 + x + x^2 + \cdots + x^{p-1} = \sum_{k=0}^{p-1} x^k \tag{3}$$

. From this follows the first rewriting

$$
\begin{aligned}
\phi_p(x) &= \sum_{k=0}^{p-1} x^k \\
&= \sum_{k=0}^{p-1} e^{k\ln(x)} \\
&= e^{0\ln(x)} + \sum_{k=1}^{p-1} e^{k\ln(x)} \\
&= 1 + \frac{e^{\ln(x)}\left(e^{(p-1)\ln(x)} - 1\right)}{e^{\ln(x)} - 1} \qquad (4)
\end{aligned}
$$

, since

$$
\sum_{k=1}^{n} e^{kA} = \frac{e^{A}\left(e^{An} - 1\right)}{e^{A} - 1}
$$

. Now we will solve this equation for p.

$$
\begin{aligned}
\phi_p(x) &= 1 + \frac{e^{\ln(x)}\left(e^{(p-1)\ln(x)} - 1\right)}{e^{\ln(x)} - 1} \\
\left(\phi_p(x) - 1\right)\left(e^{\ln(x)} - 1\right) &= e^{\ln(x)}\left(e^{(p-1)\ln(x)} - 1\right) \\
\frac{\left(\phi_p(x) - 1\right)\left(e^{e^{\ln(x)}} - 1\right)}{e^{\ln(x)}} + 1 &= e^{(p-1)\ln(x)} \\
(p-1)\ln(x) &= \ln\left(\frac{\left(\phi_p(x) - 1\right)\left(e^{\ln(x)} - 1\right)}{e^{\ln(x)}} + 1\right) \\
(p-1)\ln(x) &= \ln\left(\frac{\left(\phi_p(x) - 1\right)(x-1) + x}{x}\right) \\
\ln(x)\left((p-1) + 1\right) &= \ln\left(\left(\phi_p(x) - 1\right)(x-1) + x\right) \\
p &= \ln\left(\frac{\left(\phi_p(x) - 1\right)(x-1) + x}{x}\right) \qquad (5)
\end{aligned}
$$

Additionally, we have our second, very easy to see, connection for prime numbers

$$
\begin{aligned}
\phi_p(x) &= x^p - 1 \\
&= \sum_{k'=0}^{p} x^{k'} - \sum_{k''=0}^{p-1} x^{k''} - 1 \\
&= \phi_{p+1}(x) - \phi_p(x) - 1 \qquad (6)
\end{aligned}
$$

. I have no idea for which this can be useful, but I think it is nice to know :P.